"TUYMAADA–2016"

Senior league

First day

1. Functions f and g are defined on the set of all integers in the interval $[-100; 100]$ and take integral values. Prove that for some integral k the number of solutions of the equation

$$f(x) - g(y) = k$$

is odd.

(*A. Golovanov*)

2. The diagonals AC and BD of a cyclic quadrilateral $ABCD$ are perpendicular and meet at point P. The point Q on the segment PC is such that $AP = QC$. Prove that the perimeter of the triangle BQD is at least $2AC$.

(*A. Kuznetsov*)

3. All the sides of a right triangle with area S are rational. Prove that there exists a right triangle not equal to the original one such that all its sides are rational and its area is S.

(*S. Chan*)

4. There are 25 masks of different colours. k sages play the following game. They are shown all the masks. Then the sages agree on their strategy. After that the masks are put on them so that each sage sees the masks on the others but can not see who wears each mask and does not see his own mask. No communication is allowed. Then each of them simultaneously names one colour trying to guess the colour of his mask. Find the minimum k for which the sages can agree so that at least one of them surely guesses the colour of his mask.

(*S. Berlov*)

Second day

5. Does there exist a quadratic trinomial $f(x)$ such that $f(1/2017) = 1/2018$, $f(1/2018) = 1/2017$, and two of its coefficients are integers?

(*A. Khrabrov*)

6. Let $\sigma(n)$ denote the sum of positive divisors of a number n. A positive integer $N = 2^r b$ is given, where r and b are positive integers and b is odd. It is known that $\sigma(N) = 2N - 1$. Prove that b and $\sigma(b)$ are coprime.

(*J. Antalan, J. Dris*)

7. A point E lies on the extension of the side AD of the rectangle $ABCD$ over D. The ray EC meets the circumcircle ω of ABE at the point $F \neq E$. The rays DC and AF meet at P. H is the foot of the perpendicular drawn from C to the line ℓ going through E and parallel to AF. Prove that the line PH is tangent to ω.

(*A. Kuznetsov*)

8. Two points A and B are given in the plane. A point X is called their *preposterous midpoint* if there is a Cartesian coordinate system in the plane such that the coordinates of A and B in this system are non-negative, the abscissa of X is the geometric mean of the abscissae of A and B, and the ordinate of X is the geometric mean of the ordinates of A and B. Find the locus of all the preposterous midpoints of A and B.

(*K. Tyschuk*)

Junior league

First day

1. Functions f and g are defined on the set of all integers in the interval $[-100; 100]$ and take integral values. Prove that for some integral k the number of solutions of the equation

$$f(x) - g(y) = k$$

is odd.

(*A. Golovanov*)

2. The diagonals AC and BD of a cyclic quadrilateral $ABCD$ are perpendicular and meet at point P. The point Q on the segment PC is such that $AP = QC$. Prove that the perimeter of the triangle BQD is at least $2AC$.

(*A. Kuznetsov*)

3. Every two cities in a country are connected either by a direct bus route or by a direct plane flight. *A clique* is a set of cities such that every two of them are connected by a direct flight. *A cluque* is a set of cities such that every two of them are connected by a direct flight, and the numbers of bus routes starting in each of them are equal. *A claque* is a set of cities such that every two of them are connected by a direct flight, and the numbers of bus routes starting in all of them are different. Prove that the the number of cities in every clique does not exceed the product of the largest possible number of cities in a cluque and the largest possible number of cities in a claque.

(*P. Borg, Y. Caro, translated by K. Kokhas*)

4. All the sides of a right triangle with area S are rational. Prove that there exists a right triangle not equal to the original one such that all its sides are rational and its area is S.

(*S. Chan*)

Second day

5. BL is the bisector of an isosceles triangle ABC. A point D is chosen on the base BC and a point E is chosen on the lateral side AB so that $AE = \frac{1}{2}AL = CD$. Prove that $LE = LD$.

(*A. Kuznetsov*)

6. Let $\sigma(n)$ denote the sum of positive divisors of a number n. A positive integer $N = 2^r b$ is given, where r and b are positive integers and b is odd. It is known that $\sigma(N) = 2N - 1$. Prove that b and $\sigma(b)$ are coprime.

(*J. Antalan, J. Dris*)

7. An equilateral triangle with side 20 is divided by three series of parallel lines into 400 equilateral triangles with side 1. What maximum number of these small triangles can be crossed (internally) by one line?

(*A. Golovanov*)

8. Consider a graph with vertices A_1, A_2, $\ldots$, A_{2017}, B_1, B_2, $\ldots$, B_{2017} and edges A_iB_i, A_iA_{i+1}, B_iB_{i+17} (in cyclic numbering). Is it true that 4 cops can catch one robber on this graph for every initial position of cops and robber? (First all the cops make their moves, then robber makes his move, then again all the cops make their moves, etc. In a move, a person can stay in his/her vertex or jump to any of the neighboring vertices. Everybody knows about positions of all others. The cops can coordinate their moves. The robber is caught if after some move he shares his vertex with some cop).

(*T. Ball, R. Bell, J. Guzman, M. Hanson-Colvin, N. Schonsheck*)