Senior league

First day

1. The sequence (a_n) is defined by $a_1 = 0$,

$$a_{n+1} = \frac{a_1 + a_2 + \ldots + a_n}{n} + 1.$$

Prove that $a_{2016} > \frac{1}{2} + a_{1000}$.

(*A. Golovanov*)

2. A cube stands on one of the squares of an infinite chessboard. On each face of the cube there is an arrow pointing in one of the four directions parallel to the sides of the face. Anton looks on the cube from above and rolls it over an edge in the direction pointed by the arrow on the top face. Prove that the cube cannot cover any 5×5 square.

(*A.Chukhnov*)

3. Altitudes AA_1, BB_1, CC_1 of an acute triangle ABC meet at H. A_0, B_0, C_0 are the midpoints of BC, CA, AB respectively. Points A_2, B_2, C_2 on the segments AH, BH, HC_1 respectively are such that $\angle A_0 B_2 A_2 = \angle B_0 C_2 B_2 = \angle C_0 A_2 C_2 = 90°$. Prove that the lines AC_2, BA_2, CB_2 are concurrent.

(*A.Pastor*)

4. For each positive integer k determine the number of solutions of the equation

$$8^k = x^3 + y^3 + z^3 - 3xyz$$

in non-negative integers x, y, z, $0 \le x \le y \le z$.

(*V. Shevelev*)

Second day

5. The ratio of prime numbers p and q does not exceed 2 $(p \ne q)$. Prove that there are two consecutive positive integers such that the largest prime divisor of one of them is p and that of the other is q.

(*A. Golovanov*)

6. The numbers a, b, c, d satisfy $0 < a \le b \le d \le c$ and $a + c = b + d$. Prove that for every internal point P of a segment with length a this segment is a side of a circumscribed quadrilateral with consecutive sides a, b, c, d, such that its incircle contains P.

(*L. Emelyanov*)

7. For every x, y, $z > \frac{3}{2}$ prove the inequality

$$x^{24} + \sqrt[5]{y^{60} + z^{40}} \ge \left(x^4 y^3 + \frac{1}{3} y^2 z^2 + \frac{1}{9} x^3 z^3 \right)^2.$$

(*K. Kokhas*)

8. A connected graph is given. Prove that its vertices can be coloured blue and green and some of its edges marked so that every two vertices are connected by a path of marked edges, every marked edge connects two vertices of different colour and no two green vertices are connected by an edge of the original graph.

(*V. Dolnikov*)

First day

1. Tanya and Serezha have a heap of 2016 candies. They make moves in turn, Tanya moves first. At each move a player can eat either one candy or (if the number of candies is even at the moment) exactly half of all candies. The player that cannot move loses. Which of the players has a winning strategy?

(A. Golovanov)

2. The point D on the altitude AA_1 of an acute triangle ABC is such that $\angle BDC = 90°$; H is the orthocentre of ABC. A circle with diameter AH is constructed. Prove that the tangent drawn from B to this circle is equal to BD.

(L.Emelyanov)

3. A cube stands on one of the squares of an infinite chessboard. On each face of the cube there is an arrow pointing in one of the four directions parallel to the sides of the face. Anton looks on the cube from above and rolls it over an edge in the direction pointed by the arrow on the top face. Prove that the cube cannot cover any 5×5 square.

(A.Chukhnov)

4. Non-negative numbers a, b, c satisfy $a^2 + b^2 + c^2 \geq 3$. Prove the inequality

$$(a + b + c)^3 \geq 9(ab + bc + ca).$$

(A.Khrabrov)

Second day

5. Positive numbers are written in the squares of a 10×10 table. Frogs sit in five squares and cover the numbers in these squares. Kostya found the sum of all visible numbers and got 10. Then each frog jumped to an adjacent square and Kostya's sum changed to 10^2. Then the frogs jumped again, and the sum changed to 10^3 and so on: every new sum was 10 times greater than the previous one. What maximum sum can Kostya obtain?

(K. Kokhas)

6. Is there a positive integer $N > 10^{20}$ such that all its decimal digits are odd, the numbers of digits 1, 3, 5, 7, 9 in its decimal representation are equal, and it is divisible by each 20-digit number obtained from it by deleting digits? (Neither deleted nor remaining digits must be consecutive.)

(S. Berlov)

7. The numbers a, b, c, d satisfy $0 < a \leq b \leq d \leq c$ and $a + c = b + d$. Prove that for every internal point P of a segment with length a this segment is a side of a circumscribed quadrilateral with consecutive sides a, b, c, d, such that its incircle contains P.

(L. Emelyanov)

8. The flights map of air company $K_{r,r}$ presents several cities. Some cities are connected by a direct (two way) flight, the total number of flights is m. One must choose two non-intersecting groups of r cities each so that every city of the first group is connected by a flight with every city of the second group. Prove that number of possible choices does not exceed $2m^r$.

(D. Conlon)