“TUYMAADA–2015”

Senior league

`First day`

1. On a playing field n footballers practiced, all of them were forwards or goalkeepers. k goals were scored during the exercise in all. Prove that Don Fabio can assign numbers from 1 to n to all footballers so that for each goal the numbers of the forward and the goalkeeper differ at least by $n-k$.

(*S. Berlov*)

2. Median BD is drawn in a triangle ABC. Bisectors of the angles ABD and ACB are perpendicular. Determine the maximum possible value of the angle BAC.

(*S. Berlov*)

3. Polynomial $P(x,y)$ with real coefficients satisfies

$$P(x+2y, x+y) = P(x,y).$$

Prove that $P(x,y) = Q\big((x^2-2y^2)^2\big)$ for some polynomial $Q(t)$.

(*A. Golovanov*)

4. The number $n!$ is written in the form ab^2, where a is squarefree. Prove that for every $\varepsilon > 0$ the inequality

$$2^{(1-\varepsilon)n} < a < 2^{(1+\varepsilon)n}$$

holds for all n large enough.

(*M. Ivanov*)

`Second day`

5. A positive integer is increased by its largest proper divisor, the number obtained is increased by its largest proper divisor and so on. Prove that after several operations a number divisible by 3^{2000} is obtained.

(*A. Golovanov*)

6. Integers $0 \le b \le c \le d \le a$ are given, $a > 14$. Prove that not all positive integers n can be written in the form

$$n = x(ax+b) + y(ay+c) + z(az+d),$$

with integral x, y, z.

(*K. Kokhas*)

7. In a triangle ABC M is the midpoint of AB and O is the circumcentre. It is known that $R-r=OM$. The external bisector of angle A meets the line BC at D, and the external bisector of angle C meets the line AB at E. Determine all possible values of the angle CED.

(*D. Shiryaev*)

8. In the plane $k(k+1)/2+1$ points are marked. Some of these points are joined by non-intersecting segments that can not pass throw the marked point so that the plane is divided into several parallelograms and one infinite region. What maximum number of segments can be drawn?

(*A. Kupavsky, A. Polyansky*)

Junior league

First day

1. 100 different real numbers are given. Prove that they can be arranged in the squares of 10×10 table so that no two numbers in squares sharing a side differ by 1.

(*A. Golovanov*)

2. A positive integer is called *funny* if for some k the sum of its digits equals k and number itself is divisible by $k+1$. What maximum number of consecutive integers can be funny?

(*O. Podlipsky*)

3. Median BD is drawn in a triangle ABC. Bisectors of the angles ABD and ACB are perpendicular. Determine the maximum possible value of the angle BAC.

(*S. Berlov*)

4. Prove that there exists a positive integer n such that in the decimal representation of each of the numbers $\sqrt{n}$, $\sqrt[3]{n}$, $\sqrt[4]{n}$, $\ldots$, $\sqrt[10]{n}$ digits $2015\ldots$ stand immediately after the decimal point.

(*A. Golovanov*)

Second day

5. A positive integer is increased by its largest proper divisor, the number obtained is increased by its largest proper divisor and so on. Prove that after several operations a number divisible by 3^{2000} is obtained.

(*A. Golovanov*)

6. Does there exist an increasing sequence (a_n) of positive integers such that each positive integer occurs exactly once among the numbers $a_{n+1} - a_n$, and each positive integer not less than some number occurs exactly once among the numbers $a_{n+2} - a_n$?

(*A. Golovanov*)

7. The extenson of bisector CL of a triangle ABC meets the circumcircle of the triangle at point K. I is the incentre of the triangle. It is known that $IL = LK$. Prove that $CI = IK$.

(*D. Shiryaev*)

8. Four sages stand around a non-transparent baobab. Each of the sages wears red, blue, or green hat. A sage sees only his two neighbours. Each of them at the same time must make a guess about the colour of his hat. If at least one sage guesses correctly, the sages win. They could consult before the game started. How should they act to win?

(*K. Kokhas*)