“TUYMAADA–2014”

Senior league

First day

1. Four consecutive three-digit numbers are divided respectively by four consecutive two-digit numbers. What minimum number of different remainders can be obtained?

(*A. Golovanov*)

2. The points K and L on the side BC of a triangle ABC are such that $\angle BAK = \angle CAL = 90^\circ$. Prove that the midpoint of the altitude drawn from A, the midpoint of KL and the circumcentre of ABC are collinear.

(*A. Akopyan, S. Boev, P. Kozhevnikov*)

3. Positive numbers a, b, c satisfy $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=3$. Prove the inequality

$$\frac{1}{\sqrt{a^3+1}}+\frac{1}{\sqrt{b^3+1}}+\frac{1}{\sqrt{c^3+1}} \le \frac{3}{\sqrt{2}}.$$

(*N. Alexandrov*)

4. A $k\times\ell$ “parallelogram” is drawn on a paper with hexagonal cells (it consists of k horizontal rows of ℓ cells each; the figure gives an example of 3×4 parallelogram). In this parallelogram a set of non-intersecting sides of hexagons is chosen; it divides all the vertices into pairs. How many ways are there to do that?

(*T. Došlić*)

Second day

5. There is an even number of cards on a table; a positive integer is written on each card. Let a_k be the number of cards where k is written. It is known that

$$a_n - a_{n-1} + a_{n-2} - \ldots \ge 0$$

for each positive integer n. Prove that the cards can be divided into pairs so that the numbers in each pair differ by 1.

(*A. Golovanov*)

6. Each of n black squares and n white squares in the plane can be obtained by a translation from each other. Every two squares of different colours have a common point. Prove that there is a point belonging at least to n squares.

(*V. Dolnikov*)

7. A parallelogram $ABCD$ is given. The excircle of triangle ABC touches the side AB at L and the extension of BC at K. The line DK meets the diagonal AC at point X; the line BX meets the median CC_1 of triangle ABC at Y. Prove that the line YL, median BB_1 of triangle ABC and its bisector CC' have a common point.

(*A. Golovanov*)

8. Let positive integers a, b, c be pairwise coprime. Denote by $g(a,b,c)$ the maximum integer not representable in the form $xa+yb+zc$ with positive integral x, y, z. Prove that

$$g(a,b,c) \ge \sqrt{2abc}.$$

(*M. Ivanov*)

Junior league

`First day`

1. Given are three different primes. What maximum number of these primes can divide their sum?

(*A. Golovanov*)

2. A $k \times \ell$ "parallelogram" is drawn on a paper with hexagonal cells (it consists of k horizontal rows of ℓ cells each; the figure gives an example of 3×4 parallelogram). In this parallelogram a set of non-intersecting sides of hexagons is chosen; it divides all the vertices into pairs. How many vertical sides can be in this set?

(*T. Došlić*)

3. The points K and L on the side BC of a triangle ABC are such that $\angle BAK = \angle CAL = 90^\circ$. Prove that the midpoint of the altitude drawn from A, the midpoint of KL and the circumcentre of ABC are collinear.

(*A. Akopyan, S. Boev, P. Kozhevnikov*)

4. Positive numbers a, b, c satisfy $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 3$. Prove the inequality

$$\frac{1}{\sqrt{a^3+1}} + \frac{1}{\sqrt{b^3+1}} + \frac{1}{\sqrt{c^3+1}} \le \frac{3}{\sqrt{2}}.$$

(*N. Alexandrov*)

`Second day`

5. For two quadratic trinomials $P(x)$ and $Q(x)$ there is a linear function $\ell(x)$ such that $P(x) = Q(\ell(x))$ for all real x. How many such linear functions $\ell(x)$ can exist?

(*A. Golovanov*)

6. Radius of the circle ω_A with centre at vertex A of a triangle ABC is equal to radius of the excircle tangent to BC. The circles ω_B and ω_C are defined similarly. Prove that if two of these circles are tangent then every two of them are tangent to each other.

(*L. Emelyanov*)

7. Each of n black squares and n white squares in the plane can be obtained by a translation from each other. Every two squares of different colours have a common point. Prove that there is a point belonging at least to n squares.

(*V. Dolnikov*)

8. There are m villages on the left bank of the Lena, n villages on the right bank and one village on an island. It is known that $(m+1, n+1) > 1$. Every two villages separated by water are connected by ferry with positive integral number.

The inhabitants of each village say that all the ferries operating in their village have different numbers and these numbers form a segment of the series of integers. Prove that at least some of them are wrong.

(*K. Kokhas*)