"TUYMAADA–2013"

Senior league

First day

1. 100 heaps of stones lie on a table. Two players make moves in turn. At each move a player can remove any non-zero number of stones from at most 99 heaps. The player that cannot move loses. Determine, for each initial position, which of the players, the first or the second, has a winning strategy.

(*K. Kokhas*)

2. Points X and Y inside the rhombus $ABCD$ are such that Y is inside the convex quadrilateral $BXDC$ and $2\angle XBY = 2\angle XDY = \angle ABC$. Prove that the lines AX and CY are parallel.

(*S. Berlov*)

3. Vertices of a connected graph cannot be coloured with less than $n+1$ colours so that adjacent verices have different colours. Prove that $n(n-1)/2$ edges can be removed from the graph so that it remains connected.

(*V. Dolnikov*)

4. Prove that if x, y, z are positive and $xyz = 1$ then

$$\frac{x^3}{x^2+y} + \frac{y^3}{y^2+z} + \frac{z^3}{z^2+x} \geq \frac{3}{2}.$$

(*A. Golovanov*)

Second day

5. Prove that every polynomial of fourth degree can be represented in the form $P(Q(x)) + R(S(x))$, where P, Q, R, S are quadratic trinomials.

(*A. Golovanov*)

6. Solve the equation $p^2 - pq - q^3 = 1$ in prime numbers.

(*A. Golovanov*)

7. The points A_1, A_2, A_3, A_4 are vertices of a regular tetrahedron with edge 1. The points B_1 and B_2 lie inside the figure bounded by the plane $A_1A_2A_3$ and the spheres of radius 1 with centres A_1, A_2, A_3. Prove that $B_1B_2 < \max(B_1A_1, B_1A_2, B_1A_3, B_1A_4)$.

(*A. Kupavsky*)

8. Cards numbered from 1 to 2^n are distributed among k children, $1 \leq k \leq 2^n$, so that each child gets at least one card. Prove that the number of ways to do that is divisible by 2^{k-1} but not by 2^k.

(*M. Ivanov*)

Junior league

First day

1. 100 heaps of stones lie on a table. Two players make moves in turn. At each move a player can remove any non-zero number of stones from at most 99 heaps. The player that cannot move loses. Determine, for each initial position, which of the players, the first or the second, has a winning strategy.

(*K. Kokhas*)

2. $ABCDEF$ is a convex hexagon such that $AC \parallel DF$, $BD \parallel AE$ and $CE \parallel BF$. Prove that $AB^2 + CD^2 + EF^2 = BC^2 + DE^2 + AF^2$.

(*N. Sedrakyan*)

3. For every positive a and b prove the inequality

$$\sqrt{ab} \leq \frac{1}{3} \cdot \sqrt{\frac{a^2+b^2}{2}} + \frac{2}{3} \cdot \frac{2}{\frac{1}{a}+\frac{1}{b}}.$$

(*A. Khrabrov*)

4. Vertices of a connected graph cannot be coloured with less than $n+1$ colours so that adjacent verices have different colours. Prove that $n(n-1)/2$ edges can be removed from the graph so that it remains connected.

(*V. Dolnikov*)

Second day

5. Each face of a $7 \times 7 \times 7$ cube is divided into unit squares. What maximum number of squares can be chosen so that no two chosen squares have a common point?

(*A. Golovanov*)

6. Quadratic trinomials with positive leading coefficients are arranged in the squares of a $6\times$ table. Their 108 coefficients are all integers from -60 to 47 (each number is used once). Prove that at least in one column the sum of all the trinomials has a real root.

(*K. Kokhas and F. Petrov*)

7. Solve the equation $p^2 - pq - q^3 = 1$ in prime numbers.

(*A. Golovanov*)

8. The point A_1 on the perimeter of a convex quadrilateral $ABCD$ is such that the line AA_1 divides the quadrilateral into two parts of equal area. The points B_1, C_1, D_1 are defined similarly. Prove that the area of the convex quadrilateral with vertices A_1, B_1, C_1, D_1 is greater than a quarter of the area of $ABCD$.

(*L. Emelyanov*)