"TUYMAADA–2012"

Senior league

First day

1. Tanya and Serezha take turns putting chips in empty squares of a chessboard. Tanya starts by putting a chip in an arbitrary square. At every next move Serezha must put a chip in the column where Tanya put her last chip, and Tanya must put a chip in the row where Serezha put his last chip. The player that cannot move loses. Which of the players has a winning strategy?

(*A. Golovanov*)

2. Quadratic trinomial $P(x)$ has two real roots and satisfies the inequality

$$P(x^3 + x) \geq P(x^2 + 1)$$

for all x. Find the sum of the roots of $P(x)$.

(*A. Golovanov, M. Ivanov, K. Kokhas*)

3. A point P is chosen inside the triangle ABC so that

$$\angle PAB = \angle PCB = \frac{1}{4}(\angle A + \angle C).$$

BL is a bisector of ABC. The line PL meets the circumcircle of triangle APC at point Q. Prove that QB is the bisector of AQC.

(*S. Berlov*)

4. Let $p = 4k + 3$ be a prime, and $\frac{m}{n}$ is irreducible fraction such that

$$\frac{1}{0^2+1} + \frac{1}{1^2+1} + \ldots + \frac{1}{(p-1)^2+1} = \frac{m}{n}.$$

Prove that p divides $2m - n$.

(*A. Golovanov*)

Second day

5. Solve the equation

$$\frac{1}{n^2} - \frac{3}{2n^3} = \frac{1}{m^2}$$

in positive integers.

(*A. Golovanov*)

6. Quadrilateral $ABCD$ is cyclic and circumscribed. Its incircle touches its sides AB and CD at points X and Y, respectively. The perpendiculars to AB and CD drawn at A and D, respectively, meet at point U, those drawn at X and Y meet at point V, and, finally, those drawn at B and C meet at point W. Prove that U, V, W are collinear.

(*A. Golovanov*)

7. Positive numbers a, b, c satisfy $abc = 1$. Prove that

$$\frac{1}{2a^2+b^2+3} + \frac{1}{2b^2+c^2+3} + \frac{1}{2c^2+a^2+3} \leq \frac{1}{2}.$$

(*V. Aksenov*)

8. Integers not divisible by 2012 are arranged on the arcs of an oriented graph. We call *the weight of a vertex* the difference between the sum of numbers on the arcs going to it and that on the arcs going from it. It is known that the weight of each vertex is divisible by 2012. Prove that non-zero integers with absolute values not exceeding 2012 can be arranged on the arcs of this graph so that the weight of each vertex is zero.

(*W. Tutte*)

Junior league

`First day`

1. Tanya and Serezha take turns putting chips in empty squares of a chessboard. Tanya starts by putting a chip in an arbitrary square. At every next move Serezha must put a chip in the column where Tanya put her last chip, and Tanya must put a chip in the row where Serezha put his last chip. The player that cannot move loses. Which of the players has a winning strategy?

(*A. Golovanov*)

2. A rectangle $ABCD$ is given. Segment DK is equal to BD and lies on the half-line DC. M is the midpoint of BK. Prove that AM is the bisector of BAC.

(*S. Berlov*)

3. Prove that N^2 arbitrary different positive integers ($N > 10$) can be arranged in $N \times N$ table so that all $2N$ sums in rows and columns are different.

(*S. Volchenkov*)

4. Let $p = 1601$ (a prime) and irreducible fraction $\frac{m}{n}$ is the sum of those fractions

$$\frac{1}{0^2+1}, \quad \frac{1}{1^2+1}, \quad \ldots, \quad \frac{1}{(p-1)^2+1},$$

whose denominators are not divisible by p. Prove that p divides $2m+n$.

(*A. Golovanov*)

`Second day`

5. The vertices of a regular 2012-gon are denoted A_1, A_2, $\ldots$, A_{2012} in some order. It is known that if $k+l$ and $m+n$ leave the same remainder when divided by 2012, then the chords A_kA_l and A_mA_n have no common points. Vasya walks around the polygon and sees that the first two vertices are denoted A_1 and A_4. How is the tenth vertex denoted?

(*A. Golovanov*)

6. Solve the equation

$$\frac{1}{n^2} - \frac{3}{2n^3} = \frac{1}{m^2}$$

in positive integers.

(*A. Golovanov*)

7. A circle lies in a quadrilateral with successive sides 3, 6, 5, 8. Prove that its radius is less than 3.

(*K. Kokhas*)

8. 25 little donkeys stand in a row; the rightmost of them is Eeyore. Winnie the Pooh wants to give a balloon of one of the seven colours of rainbow to each donkey so that successive donkeys receive balloons of different colours and at least one ballon of each colour is given to somebody. Eeyore wants to give to each of 24 remaining donkeys a pot of one of six colours of rainbow (except red) so that at least one pot of each colour is given to somebody (but successive donkeys can receive pots of the same colour). Which of the friends has more ways to get his plan implemented and how many times more?

(*F. Petrov*)