“TUYMAADA–2011”

Senior league

First day

1. Red, blue, and green children stand in a circle. When a teacher asked the red children that have a green neighbour to lift a hand, 20 children lifted their hands. When she asked the blue children that have a green neighbour to lift a hand, 25 children lifted their hands. Prove that some child that lifted the hand had two green neighbours.

(*A. Golovanov*)

2. Circles ω_1 and ω_2 intersect at points A and B; M is the midpoint of AB. Points S_1 and S_2 lie on the line AB. The tangents drawn from S_1 to ω_1 touch it at X_1 and Y_1; the tangents drawn from S_2 to ω_2 touch it at X_2 and Y_2. Prove that if the line X_1X_2 passes through M, then Y_1Y_2 also passes through M.

(*A. Akopyan*)

3. At each square of an infinite chessboard the minimum number of moves allowing a knight to reach this square from a given square O is written. A square is called *singular* if 100 is written in it and 101 is written in all the squares sharing a side with it. How many singular squares are there?

(*A. Golovanov*)

4. There are exactly 10 biquadrates and 100 cubes among several consecutive positive integers. Prove that there are at least 2000 perfect squares among these positive integers.

(*A. Golovanov*)

Second day

5. All the numbers greater than 1 are coloured in two colours (both colours are used). Prove that there exist real a and b such that the numbers $a + \frac{1}{b}$ and $b + \frac{1}{a}$ have different colours.

(*A. Golovanov*)

6. In a word of more than 10 letters, every two consecutive letters are different. Prove that one can change places of two consecutive letters so that the resulting word is not periodic (that is, cannot be divided into equal subwords).

(*A. Golovanov*)

7. In a convex hexagon $AC'BA'CB'$ every two opposite sides are equal. A_1 is the point of intersection of BC with the perpendicular bisector of AA'. B_1 and C_1 are defined similarly. Prove that A_1, B_1, and C_1 are collinear.

(*A. Akopyan*)

8. $P(n)$ is a quadratic trinomial with integer coefficients. For each positive integer n the number $P(n)$ has a proper divisor d_n (i. e. $1 < d_n < P(n)$) so that the sequence (d_n) is increasing. Prove that either $P(n)$ is the product of two linear polynomials with integer coefficients or all the values of $P(n)$ are divisible by the same integer $m > 1$.

(*A. Golovanov*)

Junior league

First day

1. Red, blue, and green children stand in a circle. When a teacher asked the red children that have a green neighbour to lift a hand, 20 children lifted their hands. When she asked the blue children that have a green neighbour to lift a hand, 25 children lifted their hands. Prove that some child that lifted the hand had two green neighbours.

(*A. Golovanov*)

2. How many ways are there to cut a 11×11 square from a 2011×2011 square so that the remained part can be divided into dominoes (1×2 rectangles)?

(*S. Volchenkov*)

3. The excircle of triangle ABC touches the side AB at P and the extensions of sides AC and BC at Q and R, respectively. Prove that if the midpoint of PQ lies on the circumcircle of ABC then the midpoint of PR also lies on that circumcircle.

(*S. Berlov*)

4. Prove that among 100000 consecutive 100-digit numbers there is a number n such that the period length of the decimal expansion of $\frac{1}{n}$ is greater than 2011.

(*A. Golovanov*)

Second day

5. All the numbers greater than 1 are coloured in two colours (both colours are used). Prove that there exist real a and b such that the numbers $a+b$ and ab have different colours.

(*A. Golovanov*)

6. A circle passing through the vertices A and B of a cyclic quadrilateral $ABCD$ intersects its diagonals AC and BD at E and F, respectively. The lines AF and BC meet at point P, the lines BE and AD meet at point Q. Prove that PQ is parallel to CD.

(*A. Akopyan*)

7. In a word of more than 10 letters, every two consecutive letters are different. Prove that one can change places of two consecutive letters so that the resulting word is not periodic (that is, cannot be divided into equal subwords).

(*A. Golovanov*)

8. The Duke of Squares left to his three sons a square estate 100 by 100 miles made of ten thousand 1 by 1 mile square plots. To divide the inheritance he showed each son a point inside the estate and assigned to this son all the plots such that the distances from their centres to that point are less than the distances to the points of his brothers. In this way the whole estate has been divided between the sons. Is it true that, irrespective of the choice of points, the part assigned to each son is connected (that is, there is a path between every two of its points, never leaving this part)?

(*A. Akopyan*)