“TUYMAADA–2010”

Senior league

First day

1. Sasha and Dima play on a 100×100 board. In the beginning Sasha chooses 50 squares and puts one king on each of them. After that Dima chooses one free square and places a rook on it. Then the players make their moves in turn (Sasha begins). At each move Sasha moves each king to an adjacent square (horizontally, vertically or diagonally) and Misha moves the rook horizontally or vertically by any number of squares. The rook cannot jump over a king or capture it. Can Sasha play so that one of the kings eventually captures the rook?

(*S. Berlov*)

2. H is the orthocentre of an acute triangle ABC. A point D is chosen within the side BC. The point P is constructed so that $ADPH$ is a parallelogram. Prove that $\angle DCP < \angle BHP$.

(*S. Berlov*)

3. On a circle 2010 digits are arranged, each equal to 1, 2, or 3. It is known that for every k each of the digits appears at most $k + 10$ times in any block of $3k$ consecutive digits. Prove that there is a block of several consecutive digits where every digits appears the same number of times.

(*S. Berlov*)

4. Prove that for every real $\alpha > 0$ the number $[\alpha n^2]$ is even for infinitely many positive integers n.

(*A. Golovanov*)

Second day

5. Baron Munchhausen boasts that he knows a remarkable quadratic triniomial with positive coefficients. The trinomial itself has an integral root; if all its coefficients are increased by 1, the trinomial obtained also has an integral root; and if all its coefficients are again increased by 1, the new trinomial, too, has an integral root. Can this be true?

(*S. Berlov*)

6. A positive integer n is given. It is known that there exist 2010 consecutive positive integers such that none of them is divisible by n but their product is divisible by n. Prove that there exist 2004 consecutive positive integers such that none of them is divisible by n but their product is divisible by n.

(*S. Berlov*)

7. The extensions of sides AB and CD of a cyclic quadrilateral $ABCD$ meet at point P, and the extensions of sides AD and BC meet at point Q. Prove that the distance between the orthocentres of triangles APD and AQB is equal to the distance between the orthocentres of triangles CQD and BPC.

(*L. Emelyanov*)

8. In a country there are 4^9 schoolchildren living in four cities. At the end of the school year a state examination was held in 9 subjects. It is known that any two students have different marks at least in one subject. However, every two students from the same city got equal marks at least in one subject. Prove that there is a subject such that every two children living in the same city have equal marks in this subject.

(*F. Petrov*)

Junior league

First day

1. Sasha and Dima play on a 100×100 board. In the beginning Sasha chooses 50 squares and puts one king on each of them. After that Dima chooses one free square and places a rook on it. Then the players make their moves in turn (Sasha begins). At each move Sasha moves each king to an adjacent square (horizontally, vertically or diagonally) and Misha moves the rook horizontally or vertically by any number of squares. The rook cannot jump over a king or capture it. Can Sasha play so that one of the kings eventually captures the rook?

(*S. Berlov*)

2. H is the orthocentre of an acute triangle ABC. A point D is chosen within the side BC. The point P is constructed so that $ADPH$ is a parallelogram. Prove that $\angle BPC > \angle BAC$.

(*S. Berlov*)

3. Three different non-zero numbers are given. Every quadratic trinomial whose coefficients are these three numbers in any order has an integral root. Prove that all such trinomials have root 1.

(*A. Golovanov*)

4. On a blackboard 2010 positive integers are written. One can remove any two numbers x, y with $y > 1$ and write either $2x+1$, $y-1$ or $2x+1$, $\frac{1}{4}(y-1)$ (if 4 divides $y-1$) instead. For instance, one can remove 3 and 5 and write 7 and 4, or 7 and 1 (taking $x=3$, $y=5$), or 11, 2 (taking $x=5$, $y=3$). This was done several times; In the beginning 2006 and 2008 were removed. Prove that the original set of numbers can not appear again in the blackboard.

(*M. Antipov*)

Second day

5. A set M of real numbers contains more than one element. It is known that for every x in M at least one of the numbers $3x-2$ and $-4x+5$ also belongs to M. Prove that M is infinite.

(*A. Golovanov*)

6. A positive integer n is given. It is known that there exist five consecutive positive integers such that none of them is divisible by n but their product is divisible by n. Prove that there exist four consecutive positive integers such that none of them is divisible by n but their product is divisible by n.

(*S. Berlov*)

7. A triangle ABC is given. Perpendicular IP is drawn from its incentre I to the line passing through A parallel to BC. The tangent to the incircle parallel to BC meets the sides AB and AC at Q and R respectively. Prove that $\angle QPB = \angle RPC$.

(*V. Smykalov*)

8. There are several cities in a country. Direct one-way flights run between some cities. Prove that one can select a group A of cities such that

(i) there is no flight between any two cities of A;

(ii) from every city outside A one can reach some city of A either by direct flight or by two flights with one change.

(*V. Dolnikov*)