“TUYMAADA–2009”

Senior league

First day

1. Three real numbers are given. Fractional part of the product of every two of them is $\frac{1}{2}$. Prove that these numbers are irrational.

(*A. Golovanov*)

2. A necklace consists of 100 blue and several red beads. It is known that every segment of the necklace containing 8 blue beads contain also at least 5 red beads. What minimum number of red beads can be in the necklace?

(*A. Golovanov*)

3. On the side AB of a cyclic quadrilateral $ABCD$ there is a point X such that diagonal BD bisects CX and diagonal AC bisects DX. What is the minimum possible value of $\frac{AB}{CD}$?

(*S. Berlov*)

4. Is there a positive integer n such that among 200th digits after decimal point in the decimal representations of $\sqrt{n}$, $\sqrt{n+1}$, $\sqrt{n+2}$, $\ldots$, $\sqrt{n+999}$ every digit occurs 100 times?

(*A. Golovanov*)

Second day

5. A magician asked a spectator to think of a three-digit number $\overline{abc}$ and then to tell him the sum of numbers $\overline{acb}$, $\overline{bac}$, $\overline{bca}$, $\overline{cab}$, and $\overline{cba}$. He claims that when he knows this sum he can determine the original number. Is that so?

(*From olympiad materials*)

6. An arrangement of chips in the squares of $n \times n$ table is called *sparse* if every 2×2 square contains at most 3 chips. Serge put chips in some squares of the table (one in a square) and obtained a sparse arrangement. He noted however that if any chip is moved to any free square then the arrangement is no more sparce. For what n is this possible?

(*S. Berlov*)

7. A triangle ABC is given. Let B_1 be the reflection of B across the line AC, C_1 the reflection of C across the line AB, and O_1 the reflection of the circumcentre of ABC across the line BC. Prove that the circumcentre of ABC lies on the line AO_1.

(*A. Akopyan*)

8. Determine the maximum number h satisfying the following condition: for every $a \in [0, h]$ and every polynomial $P(x)$ of degree 99 such that $P(0) = P(1) = 0$, there exist $x_1, x_2 \in [0, 1]$ such that $P(x_1) = P(x_2)$ and $x_2 - x_1 = a$.

(*F. Petrov, D. Rostovsky, A. Khrabrov*)

Junior league

First day

1. All squares of a 20×20 table are empty. Misha and Sasha in turn put chips in free squares (Misha begins). The player after whose move there are four chips on the intersection of two rows and two columns wins. Which of the players has a winning strategy?

(*A. Golovanov*)

2. $P(x)$ is a quadratic trinomial. What maximum number of terms equal to the sum of the two preceding terms can occur in the sequence $P(1)$, $P(2)$, $P(3)$, $\ldots$?

(*A. Golovanov*)

3. In a cyclic quadrilateral $ABCD$ the sides AB and AD are equal, $CD > AB + BC$. Prove that $\angle ABC > 120°$.

(*From olympiad materials*)

4. Each of the subsets A_1, A_2, $\ldots$, A_n of a 2009-element set X contains at least 4 elements. The intersection of every two of these subsets contains at most 2 elements. Prove that in X there is a 24-element subset B containing neither of the sets A_1, A_2, $\ldots$, A_n.

(*From olympiad materials*)

Second day

5. A magician asked a spectator to think of a three-digit number $\overline{abc}$ and then to tell him the sum of numbers $\overline{acb}$, $\overline{bac}$, $\overline{bca}$, $\overline{cab}$, and $\overline{cba}$. He claims that when he knows this sum he can determine the original number. Is that so?

(*From olympiad materials*)

6. M is the midpoint of base BC in a trapezoid $ABCD$. A point P is chosen on the base AD. The line PM meets the line CD at a point Q such that C lies between Q and D. The perpendicular to the bases drawn through P meets the line BQ at K. Prove that $\angle QBC = \angle KDA$.

(*S. Berlov*)

7. An arrangement of chips in the squares of $n \times n$ table is called *sparse* if every 2×2 square contains at most 3 chips. Serge put chips in some squares of the table (one in a square) and obtained a sparse arrangement. He noted however that if any chip is moved to any free square then the arrangement is no more sparce. For what n is this possible?

(*S. Berlov*)

8. The sum of several non-negative numbers is not greater than 200, while the sum of their squares is not less than 2500. Prove that among them there are four numbers whose sum is not less than 50.

(*A. Khabrov*)