“TUYMAADA–2008”

Senior league

First day

1. Several irrational numbers are written on a blackboard. It is known that for every two numbers a and b on the blackboard, at least one of the numbers $\frac{a}{b+1}$ and $\frac{b}{a+1}$ is rational. What maximum number of irrational numbers can be on the blackboard?

(*A. Golovanov*)

2. Is it possible to arrange on a circle all composite positive integers not exceeding 10^6, so that no two neighbouring numbers are coprime?

(*A. Golovanov*)

3. Point I_1 is the reflection of incentre I of triangle ABC across the side BC. The circumcircle of BCI_1 intersects the line II_1 again at point P. It is known that P lies outside the incircle of the triangle ABC. Two tangents drawn from P to the latter circle touch it at points X and Y. Prove that the line XY contains a medial line of the triangle ABC.

(*L. Emelyanov*)

4. A group of persons is called *good* if its members can be distributed to several rooms so that nobody is acquainted with any person in the same room but it is possible to choose a person from each room so that all the chosen persons are acquainted with each other.

A group is called *perfect* if it is good and every set of its members is also good.

A perfect group planned a party. However one of its members, Vasya, brought his acquaintance Petya, who was not originally expected, and introduced him to all his other acqaintances. Prove that the new group is also perfect.

(*C. Berge*)

Second day

5. Every street in the city of Hamiltonville connects two squares, and every square may be reached by streets from every other. The governor discovered that if he closed all squares of any route not passing any square more than once, every remained square would be reachable from each other. Prove that there exists a circular route passing every square of the city exactly once.

(*S. Berlov*)

6. A set X of positive integers is called *nice* if for each pair a, $b \in X$ exactly one of the numbers $a+b$ and $|a-b|$ belongs to X (the numbers a and b may be equal). Determine the number of nice sets containing the number 2008.

(*F. Petrov*)

7. A loader has two carts. One of them can carry up to 8 kg, and another can carry up to 9 kg. A finite number of sacks with sand lie in a storehouse. It is known that their total weight is more than 17 kg, while each sack weighs not more than 1 kg. What maximum weight of sand can the loader carry on his two carts, regardless of particular weights of sacks?

(*M.Ivanov, D.Rostovsky, V.Frank*)

8. A convex hexagon is given. Let s be the sum of the lengths of the three segments connecting the midpoints of its opposite sides. Prove that there is a point in the hexagon such that the sum of its distances to the lines containing the sides of the hexagon does not exceed s.

(*N. Sedrakyan*)

Junior league

First day

1. Portraits of famous scientists hang on a wall. The scientists lived between 1600 and 2008, and none of them lived longer than 80 years. Vasya multiplied the years of birth of these scientists, and Petya multiplied the years of their death. Petya's result is exactly $\frac{5}{4}$ times greater than Vasya's result. What minimum number of portraits can be on the wall?

(*V. Frank*)

2. Prove that all composite positive integers not exceeding 10^6 may be arranged on a circle so that no two neighbouring numbers are coprime.

(*A. Golovanov*)

3. 100 unit squares of an infinite squared plane form a 10×10 square. Unit segments forming these squares are coloured in several colours. It is known that the border of every square with sides on grid lines contains segments of at most two colours. (Such square is not necessarily contained in the original 10×10 square.) What maximum number of colours may appear in this colouring?

(*S. Berlov*)

4. Point I_1 is the reflection of incentre I of triangle ABC across the side BC. The circumcircle of BCI_1 intersects the line II_1 again at point P. It is known that P lies outside the incircle of the triangle ABC. Two tangents drawn from P to the latter circle touch it at points X and Y. Prove that the line XY contains the medial line of the triangle ABC.

(*L. Emelyanov*)

Second day

5. A loader has a waggon and a little cart. The waggon can carry up to 1000 kg, and the cart can carry only up to 1 kg. A finite number of sacks with sand lie in a storehouse. It is known that their total weight is more than 1001 kg, while each sack weighs not more than 1 kg. What maximum weight of sand can the loader carry in the waggon and the cart, regardless of particular weights of sacks?

(*M.Ivanov, D.Rostovsky, V.Frank*)

6. Let $ABCD$ be an isosceles trapezoid with $AD \parallel BC$. Its diagonals AC and BD intersect at point M. Points X and Y on the segment AB are such that $AX = AM$, $BY = BM$. Let Z be the midpoint of XY and N is the point of intersection of the segments XD and YC. Prove that the line ZN is parallel to the bases of the trapezoid.

(*A. Akopyan, A. Myakishev*)

7. A set X of positive integers is called *nice* if for each a, $b \in X$ exactly one of the numbers $a + b$ and $|a - b|$ belongs to X (the numbers a and b may be equal). Determine the number of all nice sets containing the number 2008.

(*F. Petrov*)

8. 250 numbers are chosen among positive integers not exceeding 501. Prove that for every integer t there are four chosen numbers a_1, a_2, a_3, a_4, such that $a_1 + a_2 + a_3 + a_4 - t$ is divisible by 23.

(*K. Kokhas*)