“TUYMAADA–2006”

Senior league

First day

1. Seven different odd primes are given. Is it possible that the difference of 8th powers of every two of them is divisible by each of the remained numbers?

(*F. Petrov, K. Sukhov*)

2. A sequence which is infinite in both directions is called *Fibonacci-type sequence* if each of its terms is the sum of the two preceding terms. How many Fibonacci-type sequences contain two successive positive integral terms not exceeding N? (We do not distinguish between sequences differing only by a shift of indices.)

(*I.Pevzner*)

3. Points A and B are given in the plane. Line l goes through B. Consider an arbitrary circle ω touching ℓ at B so that A lies outside ω. Tangents to ω from A touch ω at X and Y. Prove that line XY passes through a fixed point independent on the choice of ω.

(*F.Bakharev*)

4. Find all the functions $f\colon (0,\infty) \to (0,\infty)$ such that $f(x+1) = f(x)+1$ and $f\left(\frac{1}{f(x)}\right) = \frac{1}{x}$ for all positive x.

(*P. Volkmann*)

Second day

5. 100 boxers of different strength participate in the Boxing Championship of Dirtytrickland. Each of them fights each other once. Several boxers formed a plot: each of them put a leaden horseshoe in his boxing-glove during one of his fights. When just one of two boxers has a horseshoe, he wins; otherwise, the stronger boxer wins. It turned out after the championship that three boxers won more fights than any of the three strongest participants. What is the minimum possible number of plotters?

(*N.Kalinin*)

6. H is the orthocentre of an acute triangle ABC and M is the point of intersection of its medians. B_1 is the midpoint of arc AC of the circumcircle of ABC. It is known that B_1M is equal to the radius of the circumcircle. Prove that $BM \geq BH$.

(*F.Bakharev*)

7. The *corner* consists of all the squares of the first row and the first column of $n \times (n-1)$ cardboard rectangle (that is, the corner contains $2n-2$ squares). All the squares of an infinite squared plane are coloured by k colours so that all the squares covered by the corner in any position are of different colour (the corner can be rotated and turned upside down). For what minimum k is it possible?

(*S. Berlov*)

8. *Set of exponents* of a positive integer is the unordered list of exponents of primes appearing in its factorization. For example, the numbers $180 = 2^2 \cdot 3^2 \cdot 5^1$ and $882 = 3^2 \cdot 2^1 \cdot 7^2$ have the same set of exponents 1, 2, 2. Two increasing arithmetical progressions (a_n) and (b_n) are such that the numbers (a_n) and (b_n) have the same set of exponents for each n. Prove that the progressions are proportional.

(*A. Golovanov*)