“TUYMAADA–2005”

Senior league

`First day`

1. All positive integers 1, 2, …, 121 are arranged in the squares of 11×11 table. Dima found the product of numbers in each row; Sasha found the product of numbers in each column. Could they get the same set of 11 numbers?

(*S. Berlov*)

2. Six members of the team of Fatalia for the International Mathematical Olympiad are selected from 13 candidates. At the Team Selection Test the candidates got a_1, a_2, …, a_{13} points ($a_i \neq a_j$ for $i \neq j$).

The Team Leader has already chosen 6 candidates and now wants to see them and nobody other in the team. With that end in view he constructs a polynomial $P(x)$ and finds the *creative potential* of each candidate by the formula $c_i = P(a_i)$. For what minimum n can he always find a polynomial $P(x)$ of degree not exceeding n such that the creative potential of all his six candidates is strictly more than that of the seven others?

(*F. Petrov, K. Sukhov*)

3. Organizers of a mathematical congress found that if they accomodate any participant in a single room the rest can be accomodated in double rooms so that two persons living in every double room know each other.

Prove that every participant can organize a round table on graph theory for himself and even number of other people so that each participant of the round table knows both his neighbours.

(*S. Berlov, S. Ivanov*)

4. In a triangle ABC, A_1, B_1, and C_1 are the points where excircles touch the sides BC, CA, and AB respectively. Prove that AA_1, BB_1, and CC_1 are sides of a triangle.

(*L. Emelyanov*)

`Second day`

5. Several rooks stand in the squares of the table shown in the figure. The rooks beat all the squares (we suppose that a rook beats the square it stands in). Prove that one can remove several rooks so that not more than 11 rooks are left and these rooks still beat all the squares.

(*D. Rostovsky, based on folklore*)

6. Given are positive integer n and infinite sequence of proper fractions

$$x_0 = \frac{a_0}{n}, \quad x_1 = \frac{a_1}{n+1}, \quad x_2 = \frac{a_2}{n+2}, \quad \ldots \qquad a_i < n + i.$$

Prove that there exist a positive integer k and integers c_1, c_2, …, c_k such that

$$c_1 x_1 + c_2 x_2 + \ldots + c_k x_k = 1.$$

(*M. Dubashinsky*)

7. I is the incentre of triangle ABC. A circle containing the points B and C meets the segments BI and CI at points P and Q respectively. It is known that $BP \cdot CQ = PI \cdot QI$. Prove that the circumcircle of the triangle PQI is tangent to the circumcircle of ABC.

(*S. Berlov*)

8. For every positive a, b, and c such that $a^2 + b^2 + c^2 = 1$, prove the inequality

$$\frac{a}{a^3 + bc} + \frac{b}{b^3 + ca} + \frac{c}{c^3 + ab} > 3.$$

(*A. Khrabrov*)