“TUYMAADA–2004”

Senior league

First day

1. Do there exist a sequence $a_1, a_2, a_3, \ldots$ of real numbers and a non-constant polynomial $P(x)$ such that $a_m + a_n = P(mn)$ for every positive integral m and n?

(*A. Golovanov*)

2. 100 straight lines are drawn in the plane so that no two of them are parallel and no three of them meet at one point. All the points of their intersection are marked. Then all the lines and k of the marked points are erased. Given the remained points of intersection, for what maximum k one can always reconstruct the original lines?

(*A. Golovanov*)

3. An acute triangle ABC is inscribed in a circle of radius 1 with centre O; all the angles of ABC are greater than 45°. B_1 is the foot of perpendicular from B to CO, B_2 is the foot of perpendicular from B_1 to AC. Similarly, C_1 is the foot of perpendicular from C to BO, C_2 is the foot of perpendicular from C_1 to AB. The lines B_1B_2 and C_1C_2 intersect at A_3. The points B_3 and C_3 are defined in the same way. Find the circumradius of triangle $A_3B_3C_3$.

(*F.Bakharev, F.Petrov*)

4. There are many opposition societies in the city of N. Each society consists of 10 members. It is known that for every 2004 societies there is a person belonging at least to 11 of them. Prove that the government can arrest 2003 people so that at least one member of each society is arrested.

(*V.Dolnikov, D.Karpov*)

Second day

5. 50 knights of King Arthur sat at the Round Table. A glass of white or red wine stood before each of them. It is known that at least one glass of red wine and at least one glass of white wine stood on the table. The king clapped his hands twice. After the first clap every knight with a glass of red wine before him took a glass from his left neighbour. After the second clap every knight with a glass of white wine (and possibly something more) before him gave this glass to the left neughbour of his left neighbour. Prove that some knight was left without wine.

(*A. Khrabrov, incorrect translation from Hungarian*)

6. The incircle of triangle ABC touches its sides AB and BC at points P and Q. The line PQ meets the circumcircle of triangle ABC at points X and Y. Find $\angle XBY$ if $\angle ABC = 90^\circ$.

(*A. Smirnov*)

7. Zeroes and ones are arranged in all the squares of $n \times n$ table. All the squares of the left column are filled by ones, and the sum of numbers in every figure of the form  (consisting of a square and its neighbours from left and from below) is even. Prove that no two rows of the table are identical.

(*O. Vanyushina*)

8. It is known that m and n are positive integers, $m > n^{n-1}$, and all the numbers $m+1$, $m+2$, $\ldots$, $m+n$ are composite. Prove that there exist such different primes p_1, p_2, $\ldots$, p_n that p_k divides $m+k$ for $k = 1, 2, \ldots, n$.

(*C. A. Grimm*)