“TUYMAADA–2003”

Senior league

First day

1. A 2003×2004 rectangle consists of unit squares. We consider rhombi formed by four diagonals of unit squares. What maximum number of such rhombi can be arranged in this rectangle so that no two of them have any common points except vertices?

2. In a quadrilateral $ABCD$ sides AB and CD are equal, $\angle A = 150^\circ$, $\angle B = 44^\circ$, $\angle C = 72^\circ$. Perpendicular bisector of the segment AD meets the side BC at point P. Find $\angle APD$.

3. Alphabet A contains n letters. S is a set of words of finite length composed of letters of A. It is known that every infinite sequence of letters of A begins with one and only one word of S. Prove that the set S is finite.

4. Find all continuous functions $f(x)$ defined for all $x > 0$ such that for every x, $y > 0$

$$f\left(x + \frac{1}{x}\right) + f\left(y + \frac{1}{y}\right) = f\left(x + \frac{1}{y}\right) + f\left(y + \frac{1}{x}\right).$$

Second day

5. Prove that for every $\alpha_1, \alpha_2, \ldots, \alpha_n$ in the interval $(0, \pi/2)$

$$\left(\frac{1}{\sin\alpha_1} + \frac{1}{\sin\alpha_2} + \ldots + \frac{1}{\sin\alpha_n}\right)\left(\frac{1}{\cos\alpha_1} + \frac{1}{\cos\alpha_2} + \ldots + \frac{1}{\cos\alpha_n}\right) \leqslant$$

$$\leqslant 2\left(\frac{1}{\sin 2\alpha_1} + \frac{1}{\sin 2\alpha_2} + \ldots + \frac{1}{\sin 2\alpha_n}\right)^2.$$

6. Which number is bigger: the number of positive integers not exceeding 1 000 000 that can be represented by the form $2x^2 - 3y^2$ with integral x and y or that of positive integers not exceeding 1 000 000 that can be represented by the form $10xy - x^2 - y^2$ with integral x and y?

7. In a convex quadrilateral $ABCD$ $AB \cdot CD = BC \cdot DA$ and $2\angle A + \angle C = 180^\circ$. Point P lies on the circumcircle of triangle ABD and is the midpoint of the arc BD not containing A. It is known that the point P lies inside the quadrilateral $ABCD$. Prove that $\angle BCA = \angle DCP$.

8. Given are polynomial $f(x)$ with non-negative integral coefficients and positive integer a. The sequence $\{a_n\}$ is defined by $a_1 = a$, $a_{n+1} = f(a_n)$. It is known that the set of primes dividing at least one of the terms of this sequence is finite. Prove that $f(x) = cx^k$ for some non-negative integral c and k.