M A T H E M A T I C S
IX International Olympiad “Tuymaada—2002”
(Senior league)

Day 1

1. Each of the points G and H lying from different sides of the plane of hexagon $ABCDEF$ is connected with all vertices of the hexagon. Is it possible to mark 18 segments thus formed by the numbers 1, 2, 3, …, 18 and arrange some real numbers at points A, B, C, D, E, F, G, H so that each segment is marked with the difference of the numbers at its ends?

(A.Golovanov)

2. The product of positive numbers a, b, c, and d is 1. Prove that

$$\frac{1+ab}{1+a}+\frac{1+bc}{1+b}+\frac{1+cd}{1+c}+\frac{1+da}{1+d}\geqslant 4.$$

(A.Khrabrov)

3. A circle having common centre with the circumcircle of triangle ABC meets the sides of the triangle at six points forming convex hexagon $A_1A_2B_1B_2C_1C_2$ (A_1 and A_2 lie on BC, B_1 and B_2 lie on AC, C_1 and C_2 lie on AB). If A_1B_1 is parallel to the bisector of angle B, prove that A_2C_2 is parallel to the bisector of angle C.

(S.Berlov)

4. A rectangular table with 2001 rows and 2002 columns is partitioned into 1×2 rectangles. It is known that any other partition of the table into 1×2 rectangles contains a rectangle belonging to the original partition. Prove that the original partition contains two successive columns covered by 2001 horizontal rectangles.

(S.Volchenkov)

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Day 2

5. A positive integer c is given. The sequence $\{p_k\}$ is constructed by the following rule: p_1 is arbitrary prime and for $k\geqslant 1$ the number p_{k+1} is any prime divisor of p_k+c not present among the numbers p_1, p_2, …, p_k. Prove that the sequence $\{p_k\}$ cannot be infinite.

(A.Golovanov)

6. Find all the functions $f(x)$, continuous on the whole real axis, such that for every real x

$$f(3x-2)\leqslant f(x)\leqslant f(2x-1).$$

(A.Golovanov)

7. The points D and E on the circumcircle of an acute triangle ABC are such that $AD=AE$. Let H be the common point of the altitudes of triangle ABC. It is known that $AH^2=BH^2+CH^2$. Prove that H lies on the segment DE.

(D.Shiryaev)

8. A real number α is given. The sequence $n_1<n_2<n_3<\ldots$ consists of all the positive integral n such that $\{n\alpha\}<\frac{1}{10}$. Prove that there are at most three different numbers among the numbers n_2-n_1, n_3-n_2, n_4-n_3, …

(A corollary of a theorem from ergodic theory)