“TUYMAADA-2001”

Senior league

First day

1. Ten volleyball teams played a tournament; every two teams met exactly once. The winner of a game gets 1 point, the loser gets 0 (there are no draws in volleyball). If the team that scored n-th has x_n points ($n = 1, \dots, 10$), prove that $x_1 + 2x_2 + \dots + 10x_{10} \geq 165$.

(*D. Teryoshin*)

2. Solve the equation

$$(a^2, b^2) + (a, bc) + (b, ac) + (c, ab) = 199.$$

in positive integers. (Here (x, y) denotes the greatest common divisor of x and y.)

(*S. Berlov*)

3. Do there exist quadratic trinomials P, Q, R such that for every integers x and y an integer z exists satusfying $P(x) + Q(y) = R(z)$?

(*A. Golovanov*)

4. Unit square $ABCD$ is divided into 10^{12} smaller squares (not necessarily equal). Prove that the sum of perimeters of all the smaller squares having common points with diagonal AC does not exceed 1500.

(*A. Kanel-Belov*)

Second day

5. All positive integers are distributed among two disjoint sets N_1 and N_2 such that no difference of two numbers belonging to the same set is a prime greater than 100.

Find all such distributions.

(*N. Sedrakyan*)

6. Non-zero numbers are arranged in $n \times n$ square ($n > 2$). Every number is exactly k times less than the sum of all the other numbers in the same cross (i.e., $2n - 2$ numbers written in the same row or column with this number). Find all possible k.

(*D. Rostovsky, A. Khrabrov, S. Berlov*)

7. $ABCD$ is a convex quadrilateral; half-lines DA and CB meet at point Q; half-lines BA and CD meet at point P. It is known that $\angle AQB = \angle APD$. The bisector of angle $\angle AQB$ meets the sides AB and CD of the quadrilateral at points X and Y, respectively; the bisector of angle $\angle APD$ meets the sides AD and BC at points Z and T, respectively. The circumcircles of triangles ZQT and XPY meet at point K inside the quadrilateral. Prove that K lies on the diagonal AC.

(*S. Berlov*)

8. Is it possible to colour all positive real numbers by 10 colours so that every two numbers with decimal representations differing in one place only are of different colours? (We suppose that there is no place in a decimal representation such that all digits starting from that place are 9's.)

(*A. Golovanov*)