“TUYMAADA-2000”

Senior league

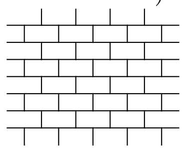
`First day`

1. Let $d(n)$ denote the number of positive divisors of n and $e(n) = \left[\frac{2000}{n}\right]$ for positive integer n. Prove that

$$d(1) + d(2) + \ldots + d(2000) = e(1) + e(2) + \ldots + e(2000).$$

2. A tangent l to the circle inscribed in a rhombus meets its sides AB and BC at points E and F respectively. Prove that the product $AE \cdot CF$ is independent of the choice of l.

3. Can the “brick wall” (infinite in all directions) drawn at the picture be made of wires of length 1, 2, 3, … (each positive integral length occurs exactly once)? (Wires can be bent but should not overlap; size of a “brick” is 1×2).



4. Prove for real x_1, x_2, …, x_n, $0 < x_k \leqslant \frac{1}{2}$, the inequality

$$\left(\frac{n}{x_1 + \ldots + x_n} - 1\right)^n \leqslant \left(\frac{1}{x_1} - 1\right) \ldots \left(\frac{1}{x_n} - 1\right).$$

`Second day`

5. Can the plane be coloured by 2000 colours so that points of all the colours present inside every circle?

6. There are 2000 cities in Graphland; some of them are connected by roads. For every city the number of roads going from it is counted. It is known that there are exactly two equal numbers among all the numbers obtained. What can be these numbers?

7. Polynomial $P(t)$ is such that for all real x

$$P(\sin x) + P(\cos x) = 1.$$

What can be the degree of this polynomial?

8. Prove that no number of the form 10^{-n}, $n \geqslant 1$, can be presented as the sum of numbers reciprocal to the factorials of diferent positive integers.