“TUYMAADA-1999”

First day

1. 1999 different planes are given in the space. Prove that there is a sphere intersecting exactly 100 of these planes.

(*V. Egorov*)

2. Find all polynomials $P(x)$ such that

$$P(x^3+1) = P(x^3) + P(x^2).$$

(*A. Golovanov*)

3. What maximum number of elements can be selected from the set $\{1, 2, 3, \ldots, 100\}$ so that no sum of any three selected numbers is equal to a selected number?

(*A. Golovanov*)

4. Prove the inequality

$$\frac{x}{y^2-z} + \frac{y}{z^2-x} + \frac{z}{x^2-y} > 1,$$

where $2 < x, y, z < 4$.

(*A. Golovanov*)

Second day

5. In the triangle ABC $\angle ABC = 100^\circ$, $\angle ACB = 65^\circ$, $M \in AB$, $N \in AC$, and $\angle MCB = 55^\circ$, $\angle NBC = 80^\circ$. Find $\angle NMC$.

(*St.Petersburg folklore*)

6. Can the graphs of a polynomial of degree 20 and the function $y = \dfrac{1}{x^{40}}$ have exactly 30 points of intersection?

(*K. Kokhas*)

7. A sequence of integers $a_0,\ a_1, \ldots a_n \ldots$ is defined by the following rules: $a_0 = 0,\ a_1 = 1,\ a_{n+1} > a_n$ for each $n \in \mathbb{N}$, and a_{n+1} is the minimum number such that no three numbers among $a_0,\ a_1, \ldots a_{n+1}$ form an arithmetical progression. Prove that $a_{2^n} = 3^n$ for each $n \in \mathbb{N}$.

(*Ya. Peredriy*)

8. A right parallelepiped (i.e. a parallelepiped one of whose edges is perpendicular to a face) is given. Its vertices have integral coordinates, and no other points with integral coordinates lie on its faces or edges. Prove that the volume of this parallelepiped is a sum of three perfect squares.

(*A. Golovanov*)